

Effective Results on non-Archimedean Tropical Discriminants

Korben Rusek *

August 29, 2012

For my adorable nephew, Sebastian Wayne Rusek, Born 6-16-2011.

Abstract

We study A -discriminants from a non-Archimedean point of view, refining earlier work on the tropical discriminant. In particular, we study the case where A is a collection of $n+m+1$ points in \mathbb{Z}^n in general position, and give an algorithm to compute the image of the A -discriminant variety under the non-Archimedean evaluation map. When $m = 2$, our approach yields tight lower and upper bounds, of order quadratic in n . We also detail a Sage package for plotting certain p -adic discriminant amoebae, and present explicit examples of point sets yielding discriminant amoebae with extremal behavior.

1 Introduction

Amoebae — the images of algebraic varieties under a valuation map — are of considerable interest in several complex variables, tropical geometry, and arithmetic dynamics [6, 8, 2]. Furthermore, in addition to applications in mathematical physics [9], amoebae have recently been used to derive efficient algorithms in real algebraic geometry and arithmetic geometry [1, 12]. In particular, the real part (and the non-Archimedean rational part) of the complement of a discriminant amoeba results in a new point of view in the classical study of discriminant complements.

In this paper, we focus on the non-Archimedean amoebae of A -discriminants, proving new complexity bounds on the topology of their closures. In particular, we exhibit some unusual behavior differing from the complex setting, and improve an earlier topological bound of Dickenstein, et. al [4].

Let A be a generic collection of $n + m + 1$ points in \mathbb{Z}^n , with $m \geq 0$. The support of a polynomial is the collection of exponent vectors with nonzero

*Partially supported by NSF MCS grant DMS-0915245. K.R. also partially supported by Sandia National Labs and DOE ASCR grant de-sc0002505. Sandia is a multiprogram laboratory operated by Sandia Corp., a Lockheed Martin Company, for the US DOE under Contract DE-AC04-94AL85000.

coefficients. The A -discriminant of the family of polynomials over a field, k , with support, A , was introduced by Gelfand, Kapranov, and Zelevinsky in their book [6]. They also discussed a simple parametric map called the Horn uniformization of this A -discriminant. We will not discuss the original parametrization further, but we will look at a dehomogenized form of it that produces the so-called reduced A -discriminant. Write $A = \{a_1, \dots, a_{n+m+1}\}$. Then let \hat{A} be the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_{n+m} & a_{n+m+1} \end{bmatrix}.$$

That is, we treat A as a matrix whose column vectors are the a_i and then we let \hat{A} be the same matrix with an extra row of all ones. Let $B = \{\beta_i\}_{i=1}^{n+m+1} \subset \mathbb{Z}^m$ be the right integer null space of \hat{A} . Then the parametric plot for the reduced A -discriminant is a map from k^{m-1} to k^m and has ℓ^{th} coordinate

$$\prod_{i=1}^{m+n+1} (\beta_{i,1}\lambda_1 + \cdots + \beta_{i,m-1}\lambda_{m-1} + \beta_{i,m})^{\beta_{i,\ell}}.$$

In our setting we will have k as a non-Archimedean valuation field, so we can take the coordinate-wise valuation. This will give us a piecewise linear object called the non-Archimedean A -discriminant amoeba.

On the other hand, we have the real semi-algebra, $(\mathbb{R}, \otimes, \oplus)$ where \otimes is standard addition and \oplus gives the minimum of two real numbers. Then given a polynomial in this semi-algebra, its zero set, a tropical varieties, is defined as the points where the graph is not differentiable. This too is a piecewise linear object. It turns out that these two families of objects, non-Archimedean amoebae and real semi-algebraic varieties have the same combinatorial type [11].

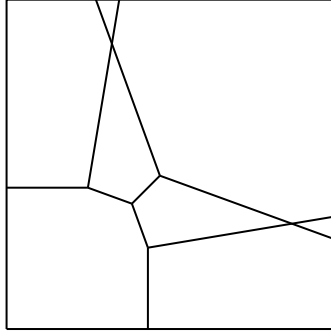
Little has been written about explicitly and efficiently representing these reduced non-Archimedean amoebae. Kapranov's non-Archimedean theorem gives a construction for non-Archimedean amoebae [7], but it requires constructing the discriminant polynomial. This can be quite inefficient. For example the family described in Example 2 has a reduced discriminant polynomial with coefficients with thousands of digits [4]. On the other hand, our software computes the amoeba in seconds. Similarly Dickenstein, Feichtner, and Sturmfels fully described the discriminant amoeba in the case where $k = \mathbb{C}\{\{t\}\}$ in [3] and Rincón built upon that same setting [5]. When k is another field, such as \mathbb{C}_p , very little has been written in the direction. In this paper, we make an effort to begin to close these gaps. We begin with an explicit representation of the reduced non-Archimedean discriminant amoeba as a collection of parametric tropical maps. Which lead themselves nicely to theorems.

Next we shift our attention to the special case $m = 2$, where the A -discriminant amoeba is a 1-dimensional object in 2-space. The process of proving the previously mentioned reduction will further lead us to reductions in the $m = 2$ case. In the real case, the best known bound on the number of components of the complement of these amoebae is $O(n^6)$ [4], but our reductions naturally lead us to an upper bound that is $O(n^2)$. Furthermore, the method leads us to ways

to produce extremal examples. That is, we exhibit a family of n -variate $n + 3$ -nomials with $O(n^2)$ connected components in the complement of the closure of their tropical A -discriminant amoebae. Finally, although it can be shown that the complex reduced A -discriminant is always solid[10] using these reductions we have found simple polynomials (discussed in Example 2) that are not simply connected. A simple example is

$$f(x, y, t) = tx^6 + \frac{44}{31}ty^6 - yt + y^6 + \frac{44}{31}tx^3 - x.$$

The associated 3-adic A -discriminant is



which has two bounded connected components in its complement—two holes. This is different than both the complex case and the standard tropical case where $k = \mathbb{C}\{\{t\}\}$.

2 Tropicalization of Parametric Non-Archimedean Maps

We can construct the reduced discriminant amoeba for the family of polynomials with support A , which will be a $m - 1$ dimensional surface in m -space. We defined A as a collection of points, but we will abuse notation and also use it to represent the n by $(n + m + 1)$ matrix whose columns are the points of A . Then $\hat{A} \in \mathbb{Z}^{(n+1) \times (n+m+1)}$ is the matrix A with an extra row of all ones added. As in the introduction, we let $B = \{\beta_{i,j} \in \mathbb{Z}^{(n+m+1) \times m}\}$ be the matrix whose columns form a basis of the right null-space of \hat{A} . Then the map, $\phi : k^{m-1} \rightarrow \mathbb{R}^m$ whose ℓ^{th} coordinate is given by

$$\sum_{i=1}^{n+m+1} \beta_{i,\ell} v(\beta_{i,1}x_1 + \cdots + \beta_{i,m-1}x_{m-1} + \beta_{i,m}), \quad (1)$$

where (β_i) are the column vectors of B , is a parametric form of the reduced discriminant amoeba[6]. Throughout we will denote this function by F , the linear forms $\beta_{i,1}x_1 + \cdots + \beta_{i,m-1}x_{m-2} + \beta_{i,m}$ by f_i , and the valuation of these

linear maps by F_i . Often we will perform a linear change of variables. This will often be denoted by a $*$ (F^*) or a subscript (F_I) when necessary.

We want to better understand the structure of the image of this particular map, ϕ . Notice that the number in front of the valuation, $\beta_{i,\ell}$, also occurs as a coefficient in the linear forms. We want to simplify ϕ in such a way that makes it easier to illustrate. To this end we will look at a slightly more general map and define a simplification. Thus the next two paragraphs will deal with a slightly more general map instead. Suppose instead that we have a map $G : k^{m-1} \rightarrow \mathbb{R}^m$, a p -adic parametric plot whose ℓ^{th} coordinate is of the form

$$\sum_{i=1}^{n+m+1} \gamma_{i,\ell} v(\beta_{i,1}x_1 + \cdots + \beta_{i,m-1}x_{m-1} + \beta_{i,m}), \quad (2)$$

where β and γ are $(n+m+1) \times m$ matrices with elements in k . Define the tropicalization of G to be the parametric tropical map, $\varphi_G : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^m$, whose ℓ^{th} coordinate map is

$$\otimes_{i=1}^{n+m+1} (v(\beta_{i,1}) \otimes r_1 \oplus \cdots \oplus v(\beta_{i,m-1}) \otimes r_{m-1} \oplus v(\beta_{i,m}))^{\gamma_{i,\ell}},$$

where the $\gamma_{i,j}$ are multiplied by the forms and we use the minimum in our semi-algebra.

Now if we let $g_i = \beta_{i,1}x_1 + \cdots + \beta_{i,m-1}x_{m-1} + \beta_{i,m}$ then we can rewrite the ℓ^{th} coordinate of G as

$$r \mapsto \sum_{i=1}^{n+m+1} \gamma_{i,\ell} v(g_i).$$

This simplification will be a key piece in our paper. For now we will just state how this is used in the process of simplifying the map ϕ , but in the next section we will fully prove the statements overviewed here. Pick $I = \{i_1, \dots, i_{m-1}\}$ with $1 \leq i_1 < \cdots < i_{m-1} \leq n+m+1$ and such that the zero set of $g_{i_1}, \dots, g_{i_{m-1}}$ consists of a unique point in k . Now, via Gauss-Jordan elimination let F_I be F under the change of variables such that $f_{i_j}^* = x_j$. Then we can write F_I in a form similar to (2). As we did with the original map, F , we can tropicalize our F_I and get ϕ_{F_I} . Let T be the collection of all such I . In the next subsection, we will prove

Theorem 1. *Let F be the parametric form of the non-Archimedean discriminant amoeba. Let T be the collection of all $I = \{i_1, \dots, i_{m-1}\} \subset \{1, \dots, n+m\}$ with $i_j < i_{j+1}$ for $j < m-1$. Let F_I be F under the linear change of variables making $f_i = \lambda_i$ and ϕ_{F_I} the tropicalization of F_I . Then we have*

$$\overline{F(k^{m-1})} = \bigcup_{I \in T} \phi_{F_I}(\mathbb{R}^{m-1}).$$

2.1 Reducing non-Archimedean Amoebae to Parametric Tropical Functions

A key difference between general non-Archimedean arithmetic and arithmetic in the real semi-algebra is the ultrametric inequality. That is, when $a, b \in \mathbb{C}_p$ then

$v(a+b) \geq \min\{v(a), v(b)\}$, whereas when a and b are elements of the real semi-algebra then $a \oplus b = \min\{a, b\}$. For example, in \mathbb{Q}_p , $v(p+p^2) = v(p) = 1$, but $v(p+(p-1)p) = v(p^2) = 2$. In the latter sum the valuation of the sum is larger than the valuation of the summands. We will call this *carrying*. This carrying makes it appear that this non-Archimedean map and the tropical maps may fail to have the same image because the carrying would possibly cause discontinuities in the image, whereas the image of the tropical map is necessarily continuous. This section will show that this does not cause a problem. That is, this section will prove theorem 1.

Since the first proof in this subsection is long and notation heavy, we will begin with an outline of the proof, so that the reader has an idea of the flow and direction of the proof beforehand. The proof begins by choosing an arbitrary parameter $\lambda \in k^{m-1}$. In the next step we apply a change of variables on the f_i . We will describe how the change of variables works. The coefficients of the f_i are taken from the rows of the B matrix. There are more rows than columns of the B matrix. Now we may select a collection $I = (i_1, \dots, i_{m-1})$ such that those rows of the matrix are linearly independent. We then perform Gauss-Jordan elimination on the *columns* of B making each i_j into a standard basis row vector. From this new matrix we can construct a new collection of functions, $\{f_1^*, \dots, f_{n+m+1}^*\}$ with $f_{i_j}^* = x_j$. These functions represent the original collection after a linear change of variables. This is the linear change of variables we will use, and the linear change of variables that results from the proof. When we apply this change of variables to F , we will call the resulting map F_I . When the correct I (based on λ) is chosen the map will have the property that it exhibits a simple parameter $z_i = p^{\ell_i}$ that approximates $F(\lambda)$. Moreover, we show that we can pick the approximating z_i in a special way such that no non-Archimedean carrying happens, on a dense subset of the domain. Since there is little carrying then we can replace the map F^* with its polarization and the parameter p^{ℓ_i} with ℓ_i . Hence we can approximate $F(\lambda)$ using one of the ϕ_{F_I} as defined earlier. Since the ϕ_{F_I} are closed maps then we will have one containment. The other containment will follow quite easily.

To further illustrate this change of variables we will go through an example. The example will explicitly construct a couple of the F_I and ϕ_{F_I} and will illustrate the difference between the images of these maps.

Example 1. *This example will take place in 2-space. The same idea presents itself in higher dimensions, but it is easier to grasp here. We will work in \mathbb{Q}_2 . In this simple example $n = 0$ and $m = 2$. Let*

$$f_1 = \lambda - 1, \quad f_2 = \lambda - 13, \quad f_3 = \lambda - 25.$$

Hence we have

$$F(\lambda) = (v(\lambda - 1) + v(\lambda - 13) + v(\lambda - 25), -v(\lambda - 1) - 13v(\lambda - 13) - 25v(\lambda - 25)).$$

Now $F_{\{1\}}$ requires $f_1^ = \lambda$, so we would have*

$$f_1^* = \lambda, \quad f_2^* = \lambda - 12, \quad f_3^* = \lambda - 24.$$

This would give us

$$\phi_{F_{\{1\}}}(r) = ((r) \otimes (r \oplus 2) \otimes (r \oplus 3), -1(r) \otimes -13(r \oplus 2) \otimes -25(r \oplus 3)).$$

(Note that the linear forms change but the coefficients in front of them do not.) Now when $\ell \neq 2, 3$ then $F_1^*(2^\ell) = \ell$, $F_2^*(2^\ell) = \min\{\ell, 2\}$, $F_3^*(2^\ell) = \min\{\ell, 3\}$, so when $\ell \neq 2, 3$ we have

$$F^*(2^\ell) = (\ell + \min\{\ell, 2\} + \min\{\ell, 3\}, -1\ell - 13\min\{\ell, 2\} - 25\min\{\ell, 3\}).$$

But F^* has a discontinuity at $\ell = 3$. That is, $F_3^*(2^3) = v(16) = 4$, but is less than or equal to 3 everywhere else, so the point $F^*(2^3) = (9, -129)$ is an isolated point when restricting to 2^ℓ . (A similar problem results from $\ell = 2$.) On the other hand $\phi_{F_{\{1\}}}$ does not have this discontinuity. The map $\phi_{F_{\{1\}}}$ does not have a carry and $\phi_{F_{\{1\}}}(3) = (8, -78)$, so this value, $(9, -129)$, appears to be lost. Furthermore, it appears this point, $(9, -129)$, will be an isolated point in the image and is not part of the images of the $\phi_{F_{\{i\}}}$. What has happened here? When we instead apply the change of variables used in making $\phi_{F_{\{3\}}}$ we see what happens. That is

$$f'_1 = \lambda + 24, f'_2 = \lambda + 12, f'_3 = \lambda$$

and

$$\phi_{F_{\{2\}}}(r) = ((r \oplus 3) \otimes (r \oplus 2) \otimes (r), -1(r \oplus 3) - 13 \otimes (r \oplus 2) \otimes -25(r)).$$

Now when $r = 4$ we have $\phi_{F_{\{2\}}}(4) = (9, -129)$. Thus we see an illustration of how though the $F_{\{i\}}$ do not have the exact same image as their corresponding $\phi_{F_{\{i\}}}$, but the collection still contains the desired values.

We will now prove that $F(\lambda)$ can be approximated as desired.

Theorem 2. Let k be an algebraically closed complete valuation field with $\mathbb{Q} \subseteq \text{ord}(k) \subseteq \mathbb{R}$. Pick any $\lambda = (\lambda_1, \dots, \lambda_{m-1}) \in k^{m-1}$ where $F(\lambda)$ is well-defined. That is, $f_i(\lambda) \neq 0$ for all i . Let $\omega \in k$ be any element with $\text{ord}(k) = 1$ (we would naturally select $\omega = t$ and $\omega = p$, for $k = \mathbb{C}\{\{t\}\}$ and $k = \mathbb{C}_p$, respectively). For any $\varepsilon > 0$ there is $I = (i_1, \dots, i_{m-1}) \in \{1, \dots, n + m + 1\}^{m-1}$ and an F^* that is F under a linear change of variables (Gauss-Jordan Elimination) such that $F_{i_j}^*(x) = x_j$ such that there are $\ell_1, \dots, \ell_{m-1} \in \mathbb{Q} \setminus \mathbb{Z}$ with $\ell_i - \ell_j \notin \mathbb{Z}$ for $i \neq j$ with $|F^*(\omega^{\ell_1}, \dots, \omega^{\ell_{m-1}}) - F(\lambda)| < \varepsilon$.

Proof. Pick $\lambda \in k^{m-1}$ as described in the hypothesis and choose any $0 < \varepsilon < 1$. For any $z \in k$ let $\lambda_z = (\lambda_1 + z, \lambda_2, \dots, \lambda_{m-1})$. Hence $f_i(\lambda_z) = f_i(\lambda) + \beta_{i,1}z$. Now I claim that for each i with $\beta_{i,1} \neq 0$, there is an N_i such that for any $z \in k$ with $v(z) > N_i$ then $F_i(\lambda_z) = F_i(\lambda)$. On the other hand, for this same i , if $v(z) < N_i$ then $F_i(\lambda_z) = v(z) + v(\beta_{i,1})$. Indeed, if for any i , we pick $N_i = F_i(\lambda) - v(\beta_{i,1})$ we have

$$\begin{aligned} F_i(\lambda_z) &\geq \min\{v(f_i(\lambda)), v(z) + v(\beta_{i,1})\} \\ &\geq \min\{N_i + v(\beta_{i,1}), v(z) + v(\beta_{i,1})\}. \end{aligned}$$

Now when $v(z) \neq N_i$ then this ultrametric inequality becomes an equality. Hence when $v(z) < N_i$ then we have $F_i(\lambda_z) = v(z) + v(\beta_{i,1})$ and when $v(z) > N_i$ we have $F_i(\lambda_z) = N_i + v(\beta_{i,1}) = F_i(\lambda)$. Now at least one i should exist with $\beta_{i,1} \neq 0$ or x_1 plays no role in any of our equations. Without loss of generality assume that 1 be the (an) index associated with the maximum such N_i and let N_1 be the relevant value. This will be our i_1 .

To show that this works as our i_1 , we pick $\ell \in \mathbb{Q}$ such that $0 < N_1 - \ell < \varepsilon$. Now if for all i , we let f'_i be f_i under the change of variables $x_1 \mapsto x_1 - \frac{\beta_{1,2}}{\beta_{1,1}}x_2 - \dots - \frac{\beta_{1,m-1}}{\beta_{1,1}}x_{m-1} - \frac{\beta_{1,m}}{\beta_{1,1}}$. Then we have

$$\begin{aligned} f'_i(\omega^\ell, \lambda_2, \dots, \lambda_{m-1}) &= \left(\beta_{i,1}\omega^\ell - \frac{\beta_{i,1}\beta_{1,2}}{\beta_{1,1}}\lambda_2 - \dots - \frac{\beta_{i,1}\beta_{1,m-1}}{\beta_{1,1}}\lambda_{m-1} - \frac{\beta_{i,1}\beta_{1,m}}{\beta_{1,1}} \right) \\ &\quad + \beta_{i,2}\lambda_2 + \dots + \beta_{i,m-1}\lambda_{m-1} + \beta_{i,m} \\ &= \beta_{i,1}\omega^\ell - \left(\frac{\beta_{i,1}\beta_{1,1}}{\beta_{1,1}}\lambda_1 + \frac{\beta_{i,1}\beta_{1,2}}{\beta_{1,1}}\lambda_2 + \dots + \frac{\beta_{i,1}\beta_{1,m-1}}{\beta_{1,1}}\lambda_{m-1} + \frac{\beta_{i,1}\beta_{1,m}}{\beta_{1,1}} \right) \\ &\quad + (\beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_2 + \dots + \beta_{i,m-1}\lambda_{m-1} + \beta_{i,m}) \\ &= \beta_{i,1}\omega^\ell - \frac{\beta_{i,1}}{\beta_{1,1}}f_1 + f_i \end{aligned}$$

We desire to know the valuation of $f'_i(\omega^\ell, \lambda_2, \dots, \lambda_{m-1})$. To this end, we look at the valuation of the three pieces in this sum. The first piece gives $v(\beta_{i,1}\omega^\ell) = v(\beta_{i,1}) + \ell$. The second, $v(\frac{\beta_{i,1}}{\beta_{1,1}}f_1) = N_1 + v(\beta_{1,1}) + v(\beta_{i,1}) - v(\beta_{1,1}) = N_1 + v(\beta_{i,1})$, since we have chosen N_1 such that $v(f_1) = N_1 + v(\beta_{1,1})$. Similarly, the third piece gives $v(f_i) = N_i + v(\beta_{i,1})$. By the ultrametric inequality, if these three items have different valuation then their sum has valuation that is the minimum of the three. Therefore if $N_i < N_1$ then $N_i < \ell$ and $F'_i = F_i$. Otherwise, if $N_i = N_1$ (it cannot be larger) then $\ell < N_1 = N_i$, which tells us that $F'_i = \ell + v(\beta_{i,1})$ and so $|F'_i - F_i| \leq \varepsilon$. We see that the latter two pieces, f_i and $\frac{\beta_{i,1}}{\beta_{1,1}}f_1$ have the same coefficient on x_1 so the difference is independent of x_1 . That is, the other F'_i are independent of x_1 , and changing the value of ℓ doesn't affect their difference. Hence we can make another change of variables sending x_1 to $\frac{x_1}{\beta_{1,1}}$ and use an ℓ that is $v\beta_1$ larger. This makes $f'_1 = x_1$, as desired. Then $i_1 = 1$ and $\ell_1 = \ell$. We can go through this procedure again with x_2 and the modified collection $\{f'_i\}$. It is clear that the newly selected i_2 will not be 1, because we have made f'_1 depend only on x_1 . We can continue this iteratively through all the variables. Each time one of the f_i will necessarily be chosen or the parametric function is under determined. Selecting i_1, \dots, i_{m-1} and $\ell_1, \dots, \ell_{m-1}$ and a final collection of $f_{i_{i_1}}^*, \dots, f_{i_{m-1}}^*$. Then the final linear forms $\{f_i^*\}$ clearly do what we want and $F^*(p^{\ell_1}, \dots, p^{\ell_{m-1}})$ approximates $F(\lambda)$ to within $b\varepsilon$ where b depends only on the original matrix, B . \square

It is again worth noting that the linear change of variables used was Gauss-Jordan elimination on the $f_{i_1}, \dots, f_{i_{m-1}}$. This means for a given collection, if we sort the i_1, \dots, i_{m-1} there is a well-defined change of variables to use together

with a finite collection of choices on the whole. Now theorem 1 will follow quite easily:

Theorem 1. *Let $T = \{I = (i_1, i_2, \dots, i_{m-1}) \mid 0 \leq i_1 < i_2 < \dots < i_{m-1} \leq n + m + 1 \text{ with } \#Z(\{f_{i_j}\}) < \infty\}$. Then*

$$\overline{F(k^{m-1})} = \bigcup_{I \in T} \varphi_{F_I}(\mathbb{R}^{m-1}).$$

Proof. Theorem 2 says that for any $z \in k^{m-1}$ there are an I as described above and an $\ell \in \mathbb{Q}^{m-1}$ such that $|\varphi_{F_I}(\ell) - F(z)| < \varepsilon$. Thus since the φ_{F_I} are closed maps and there are finitely many of them then $\overline{F(k^{m-1})} \subset \bigcup \varphi_{F_I}(\mathbb{R}^{m-1})$.

Now it suffices to show that $F(k^{m-1})$ is dense in $\varphi_I(\mathbb{R}^{m-1})$. Pick any I with $Z(f_I)$ finite and any $\ell \in \mathbb{Q}^{m-1}$. We may assume that $\ell_i - \ell_j \notin \mathbb{Z}$ for $i \neq j$ and $\ell_i \notin \mathbb{Z}$. Clearly the collection of all such $\ell_1, \dots, \ell_{m-1}$ is still dense in \mathbb{R}^{m-1} . Let F^* be F under the linear change of variables making $f_{i_j} = x_j$, as described in the previous theorem. Then it is clear that

$$F_i^*(\omega^{\ell_1}, \dots, \omega^{\ell_{m-1}}) = \min_j \{v(a_{i,j}^* \omega^{\ell_j})\},$$

because no carrying can occur since $\ell_i - \ell_j \notin \mathbb{Z}$. Therefore $F^*(\omega_1^\ell, \dots, \omega_{m-1}^\ell) = \varphi_I(\ell_1, \dots, \ell_{m-1})$ and so $\overline{F(k^{m-1})} \supset \bigcup \varphi_{F_I}(\mathbb{R}^{m-1})$ as desired. \square

3 The Case $m = 2$ and $k = \mathbb{Q}_p$

When we apply the change of variables in the main theorem we are essentially approximating the zero of a collection of $m - 1$ linear forms. We see this because as ℓ_i goes to infinity the p -adic parameter p^{ℓ_i} goes to zero and hence so do the associated linear forms. In the special case where $m = 2$ this is exactly what we are doing because there is only one parameter. To simplify notation in this section let a and b in \mathbb{R}^{n+m-1} be the columns of B , and let ϕ_i be the polarization of F after applying the variable change making $f_i = \lambda$. Given a particular i where $a_i \neq 0$, we have $f_i = a_i \lambda + b_i$ and the desired change of variables producing ϕ_i is $\lambda \mapsto \frac{p^\ell - b_i}{a_i}$. As ℓ goes to infinity, λ approaches $\frac{-b_i}{a_i}$, namely, the zero of f_i .

Let $z_i = \frac{-b_i}{a_i}$. Now if $z_i \equiv z_j \pmod{p^q}$ then $\phi_i(\ell) = \phi_j(\ell)$ for $\ell \leq q$ because we are using strict minimum. Remember that $F_i := v(f_i)$. When $z_i \equiv z_j \pmod{p^q}$ then $F_i(z_j + p^\ell) = \ell + v(a_i)$ for $\ell < q$, because $F_i(z_j) \geq q + v(a_i) > \ell + v(a_i)$. On the other hand for $\ell > q$ we have that $F_i(z_j + p^\ell)$ is the constant $q + v(a_i)$. These facts mean that for a particular i , ϕ_i is linear (or constant) everywhere except at the collection of points $v(z_i - z_j)$ for $j \neq i$. In the language of tropical geometry, we are saying that $V = \{v(z_i - z_j)\}_j$ is the tropical hypersurface of the parametric plot ϕ_i .

This tells us that there is overlap between the various ϕ_i for various values of the parameter ℓ . With this in mind we will create a tree mapping out the possible differences. A nonzero element $a \in \mathbb{Q}_p^\times$ with $v(a) = j$ can be written

$a = \sum_{i \geq j} a_i p^i$ with $a_i \in \mathbb{Z}/p\mathbb{Z}$ and $a_j \neq 0$. For a given $a = \sum_{i \geq j} a_i p^i$, we will call a_i the *digit* of a at i . If $i < j$ then the digit of a at i is defined to be zero. Thus we can construct a tree expressing the relationships between these elements. The head node represents the smallest value in $\{v(z_i - z_q) \mid i \neq q\}$. If j is this value then the tree will have a branch for each distinct digit at j among all the z_i . Now each branch represents a different digit at j , and we associate the elements with that digit at j to the branch associated to that digit at j . For a given branch we follow the same procedure, but only with the z_i associated to that branch. We repeat this iteratively and eventually a branch will have only a single element associated to it. We add one more node to the end of this branch and label it with the given z_i .

Now these two paragraphs together tell us that a given branch, between two non-leaf nodes, represents a line segment in the plot of the amoeba. We also see that every element associated to that branch contributes the value of the coefficient multiplied by its linear form to the slope of the branch. This is because these are precisely the forms producing the parameter plus a constant and the other elements are the ones producing a simple constant. The branches connected to leaf nodes, on the other hand, still have slope that is the value of the number multiplied by the associated linear form, but they represent a ray, because now the parameter, ℓ , goes to infinity. Finally, there is possibly one more ray. When the parameter, ℓ , goes to negative infinity, then we can have another ray. This only happens when the coefficients on the linear forms don't add to zero. Though we are assuming the a_i and the b_i add to zero, we can get a nonzero sum in the case that one of the a_i is zero, because then the relevant b_i will not be used in the sum. We give an example to illustrate what we've mentioned.

Example 2. *Consider the support*

$$\begin{bmatrix} 6 & 0 & 0 & 0 & 3 & 1 \\ 0 & 3 & 1 & 6 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

This is the support of the so-called Rusek-Shih example

$$f(x, y, t) = tx^6 + \frac{44}{31}ty^6 - yt + y^6 + \frac{44}{31}tx^3 - x$$

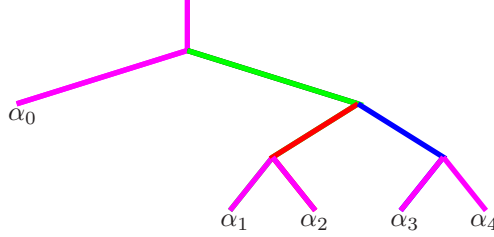
from [4]. We then choose our B matrix to be the transpose of the following:

$$\begin{bmatrix} -2 & 35 & -33 & -12 & 0 & 12 \\ 2 & -11 & 9 & 4 & -4 & 0 \end{bmatrix}$$

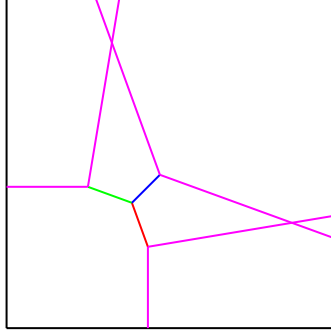
Here is an example tree. If our z_i are $1, 11/35, 3/11, 1/3$, and 0 then we have the following 3-adic expansions starting from index -1 .

$$\begin{aligned} \alpha_0 &= \frac{1}{3} &= [1, 0, 0, 0, \dots] \\ \alpha_1 &= \frac{11}{35} &= [0, 1, 2, 2, \dots] \\ \alpha_2 &= \frac{1}{11} &= [0, 1, 0, 0, \dots] \\ \alpha_3 &= \frac{3}{11} &= [0, 0, 2, 1, \dots] \\ \alpha_4 &= 0 &= [0, 0, 0, 0, \dots] \end{aligned}$$

These elements would then be put into the previously described tree as



Notice that the elements to the left of the first node on the right begin with $[0, 1]$ while the ones to the right begin with $[0, 0]$ and similar relations can be seen below the other nodes. Each branch of this tree represents a segment (or ray, for the leaves) of constant slope. In particular, with this example, when we plot the amoeba we get



The colors are there to help indicate which branch corresponds to which segment or ray. Notice the node in the middle of the graph that is not connected to any ray. This corresponds to the node earlier mentioned in the graph, because it has no leafs as direct descendants. Also, there is an extra ray. This accounts for ℓ approaching negative infinity and the fact that one of the last two linear forms has no zero.

This p -nary tree we have constructed has $n + 3$ leaves, when none of the a_i are zero, otherwise we include the extra “leaf” on top like our previous example. It is a basic fact of graph theory that such a tree will have no more than $2n + 4$ branches, when $n > 1$. As explained earlier each branch represents a ray or a segment in the graph of the p -adic amoeba. That is, each branch represents a straight piece of the amoeba. This fact will give us an upper bound on the number of connected components of the complement of the amoeba. If we replace each segment or ray with an entire line then we have a line arrangement. It is then well known that such a line arrangement has no more than $\binom{2n+4}{2} + \binom{2n+4}{1} + \binom{2n+4}{0} = 2n^2 + 9n + 11$ chambers in its complement. Therefore the p -adic amoeba also has no more than $2n^2 + 9n + 11$ complement components when none of the a_i are zero. On the other hand, when an a_i is zero we have fewer nodes and hence 2 fewer branches, but we have one extra ray accounting for $\ell \rightarrow -\infty$. Thus the same bound still applies.

It seems unlikely that this is a strict upper bound, but it is not too difficult to generate examples that have $O(n^2)$ examples. That is, we can show that the bound is asymptotically tight. We write this as a theorem and in section 4 we will look at a family of examples exhibiting the asymptotic bound.

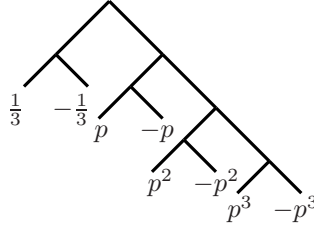
Theorem 3. *The closure of reduced A -discriminant of $n + 3$ points in general position has no more than $2n^2 + 9n + 11$ complement components. Moreover, Section 4 evinces supports A_n in \mathbb{Z}^n with cardinality $n + 3$ and primes p_n such that the p_n -adic discriminant amoeba has quadratically (quadratic in n) many connected components in its complement.*

4 Extremal p -adic Family

We will construct a family of A matrices admitting quadratically many complement components. We begin by constructing a B matrix that satisfies our requirements and then work backwards from there to get the A matrix. Let p be a prime number and let k be any integer larger than 2. We define $D \in \mathbb{Z}^{2 \times (2k+2)}$, by $D_{2,2i} = p^{i-1}$ and $D_{2,2i+1} = -p^{i-1}$ for $i = 1, \dots, k+1$ and $D_{1,1} = D_{1,2} = -k$ and $D_{1,j} = 1$ for $j > 2$. That is, D has the form:

$$\begin{bmatrix} -k & -k & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & -p & p & -p^2 & p^2 & \cdots & -p^k & p^k \end{bmatrix}.$$

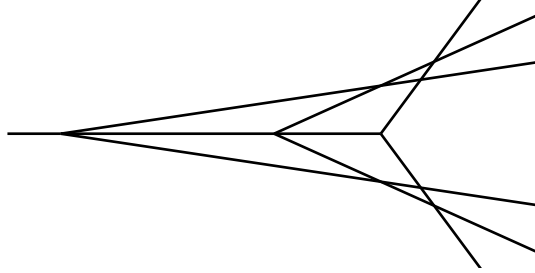
Our B matrix would be the transpose of D . Now the zeros of our linear forms are $\pm \frac{1}{k}, \pm p, \pm p^2, \dots, \pm p^n$. The p -adic order of the first two elements is less or equal to 0 and the others are their respective exponents on p . Therefore the associated tree is rather easy to form. For example, for $k = 3$ and $p = 2$ we have:



For larger k the tree extends further to the right, whereas for larger p the leaves for p^i and $-p^i$ branch directly from the main branch on the right rather than having their own mutual branch first. This is because the i^{th} digits for p^i and $-p^i$ is the same only when $p = 2$. For example, $9 = 0 \cdot 3 + 1 \cdot 3^2$, $3 = 1 \cdot 3$, and $-3 = 2 \cdot 3 + 3 \cdot 3^2 + 3 \cdot 3^3 + \dots$, while $4 = 0 \cdot 2 + 1 \cdot 2^2$, $2 = 1 \cdot 2$, and $-2 = 1 \cdot 2 + 1 \cdot 2^2 + \dots$. That is, $\text{ord}_3(3 - (-3)) = 1 = \text{ord}_3(3 - 0)$, while $\text{ord}_2(2 - (-2)) = 2$.

The slope of a branch is the sum of the (a_i, b_i) of the z_i associated to that branch. Therefore any non-leaf branch on the right has a slope of the form

$(m, 0)$, because each p^i will cancel out with its matching $-p^i$, but the 1's in the first coordinate will add. Now at the branch point between p^i and $-p^i$ we have rays in the direction $(1, p^i)$ and $(1, -p^i)$. That is, we have a line in the positive x -direction with rays emanating with slopes $\pm p^i$. Furthermore each successive ray in the direction $(1, p^i)$ is further along the x -axis than the previous one because its associated branch in the tree splits further along the main branch. Therefore, because the slope is less steep, the ray for p^i (resp. $-p^i$) intersects the ray for p^j (resp. $-p^j$) for all $j > i$. The points from which these rays are emanating are independent of p . Thus for each k , a p can be chosen assuring the intersections are non-degenerate. That is, for large enough p the p^i ray intersects the p^j with a smaller x -coordinate than the starting position of the p^{j+1} ray. Hence the rays above the x -axis give a line arrangement with at least $\binom{k}{2} + \binom{k}{1} + 1$ components. Similarly the rays below the x -axis give the same number of components, except one of these components on the right is already accounted for in the previous count. This gives us at least $k^2 + k + 1$ components in the complement of the amoeba. As a visual example, here is the relevant part of the discriminant amoeba for $p = 3$ and $k = 3$. You can also see the far right chamber that is not cut into two.



Now $k = \frac{n+m-2}{2}$. Hence the number of components is quadratic in n .

Now constructing an A matrix to accompany such a B matrix is not hard. First we find the null space, N , of D . It will be a $2k$ by $2k + 2$ matrix. For $i = 1, \dots, k$, the odd rows will be $N(2i - 1, 1) = -kp^i + 1$ and $N(2i - 1, 2) = kp^i + 1$ and $N(2i - 1, 2i + 1) = 2k$, all other coordinates of that row being zero. Similarly, for $i = 1, \dots, k$, the even rows will be $N(2i, 1) = kp^i + 1$, $N(2i, 2) = -kp^i + 1$, and $N(2i, 2i + 2) = 2k$, again all other coordinates being zero. It is clear that the rows of N are linearly independent and it is the proper dimension. A small bit of arithmetic verifies that the rows of N are orthogonal to the columns of B . Finally we can remove any row of N to get our desired A matrix. We remove one row because B should be the null space of \hat{A} rather than the null space of

A. Therefore A can have the form

$$\begin{bmatrix} -kp+1 & kp+1 & 2k & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ kp+1 & -kp+1 & 0 & 2k & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -kp^2+1 & kp^2+1 & 0 & 0 & 2k & 0 & \cdots & 0 & 0 & 0 & 0 \\ kp^2+1 & -kp^2+1 & 0 & 0 & 0 & 2k & \cdots & 0 & 0 & 0 & 0 \\ & & & & & & \cdots & & & & \\ -kp^{k-1}+1 & kp^{k-1}+1 & 0 & 0 & 0 & 0 & \cdots & 2k & 0 & 0 & 0 \\ kp^{k-1}+1 & -kp^{k-1}+1 & 0 & 0 & 0 & 0 & \cdots & 0 & 2k & 0 & 0 \\ -kp^k+1 & kp^k+1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2k & 0 \end{bmatrix}$$

5 Sage Code

When $m = 2$ this easily lends itself to a simple algorithm. A Sage package can be found at “<http://math.tamu.edu/~krusek/pamoeba.sage>”. One inputs the B matrix and the p -adic field to use and line objects are returned that represent the segments and rays of the p -adic amoeba. The code itself constructs the tree described in section 3 then for each branch it creates a sage line object representing the image of that branch. Here is a short code snippet to plot the extremal example from the previous section with $k = 3$ and $p = 3$.

```
load "pamoeba.sage"
B = MatrixSpace(ZZ,2,8)([[-3,-3,1,1,1,1,1,1],[-1,1,-3,3,-9,9,-27,27]])
B = B.transpose()
K = Qp(3)
lms = getamoeba2(B,K)
show(sum(lms))
```

References

- [1] Martin Avendaño, Ashraf Ibrahim, J. Maurice Rojas, and Korben Rusek. Faster p -adic feasibility for certain multivariate sparse polynomials. *Journal of Symbolic Computation*, 2011. To appear.
- [2] Matthew Baker and Robert Rumely. Equidistribution of small points, rational dynamics, and potential theory. *Ann. Inst. Fourier (Grenoble)*, 56(3):625–688, 2006.
- [3] Alicia Dickenstein, Eva Maria Feichtner, and Bernd Sturmfels. Tropical discriminants. *The Journal of the American Mathematical Society*, 20:1111–1133, 2007.
- [4] Alicia Dickenstein, J. Maurice Rojas, Korben Rusek, and Justin Shih. Extremal real algebraic geometry and a -discriminants. *Moscow Mathematical Journal*, 7(3):425–452, 2007.
- [5] Rincón Filipe. Computing tropical linear space. *To appear in the Journal of Symbolic Computation*, page 15, 2012.

- [6] Israel Moseyevitch Gel'fand, Misha M. Kapranov, and Andrei V Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants*. Mathematics: Theory & Applications. Birkhäuser, Boston, 1994.
- [7] Mikhail M. Kapranov. Amoebas over non-archimedean fields. *University of Toronto*, 2000.
- [8] Lisa Nilson and Mikael Passare. Discriminant coamoebas in dimension two. *pre-print*, page 17, 2009.
- [9] Mikael Passare, Dmitry Pochekutov, and August Tsikh. Amoebas of complex hypersurfaces in statistical thermodynamics. *pre-print*, page 18, 2011.
- [10] Mikael Passare, Timur Sadykov, and August Tsikh. Singularities of hypergeometric functions in several variables. *Compos. Math.*, 141(3):787–810, 2005.
- [11] Mikael Passare and August Tsikh. Amoebas: Their spines and their contours. *Contemporary Math*, 377:275–288, 2005.
- [12] Philippe Pebay, J. Maurice Rojas, and David Thompson. Optimizing n -variate $(n+k)$ -nomials for small k . *Theoretical Computer Science, Symbolic-Numeric Computation*, 412(16):1457–1469, 2011. 2009 Special Issue.